**Worksheet for Teaching Module**  
**Iterations, Fractals and Chaos (Lesson 1)**

**Topic: Iterations and Fractals**

<table>
<thead>
<tr>
<th>Equipment(s) needed for each group of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>✓ 5 to 10 transparencies</td>
</tr>
<tr>
<td>✓ Rulers</td>
</tr>
<tr>
<td>✓ Marking pens (which can be used to write on transparencies)</td>
</tr>
<tr>
<td>✓ A die</td>
</tr>
<tr>
<td>✓ 1 computer with internet connection</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Equipment(s) needed for the whole class</th>
</tr>
</thead>
<tbody>
<tr>
<td>✓ An overhead projector</td>
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</tbody>
</table>

**Introduction**

In this lesson, we shall study some basic properties of a fractal, such as self-similarity and its dimension, using the Sierpinski Triangle as our example.

**Part 1 (The Random Walk Game)**

Play the following game and answer the questions after discussing with your groupmates.

**The Game (Part A)**

Play the Random Walk Game on a transparency. The steps are as follows.

i. Go to the webpage [http://www.mathdb.org/](http://www.mathdb.org/), click into Teaching Modules > Fractals, Iterations and Chaos > Lesson 1 > Activity 1A and print several copies of the transparency provided there. On each transparency a large equilateral triangle is drawn. Colour the vertices as red, blue and green clockwise.

ii. Choose any point inside the triangle as the starting point and mark it as a small dot using a marking pen.
iii. Define the numbers 1 and 2 as red, 3 and 4 as blue and 5 and 6 as green. Roll a die and if the number obtained is a red number, mark the midpoint between the starting point and the red vertex and join the starting point and that midpoint. For the number is a blue or green number, choose the blue or green vertex respectively.

iv. Take the midpoint obtained in Step iii as the new starting point and repeat Step iii.

v. Repeat Steps iii and iv for at least 8 points.

1. Do you think that the lines are drawn randomly, without any pattern?

2. If the process continues, do you think the mid-points obtained would cover the whole triangle?

The Game (Part B)

vi. Repeat the game in Part A, but this time do not join the line between the starting point and midpoint. Mark the new midpoint only. Repeat the process for at least 20 points.

3. Do you think that the points are marked randomly without any pattern?

Go to the webpage http://www.mathdb.org/, click into Teaching Modules > Fractals, Iterations and Chaos > Lesson 1 > Activity 1B and repeat the game with more points. You may try with more than 10000 points.

4. Do you still think that the points are marked randomly without any pattern?

The Game (Part C)

vii. Repeat the game in Part B on five to ten transparencies with different starting points.
5. Do the points on different transparencies show similar patterns or are they completely different and random?

The Game (Part D)

viii. Place the transparencies together, with the vertices matching in their colours, and view them using an overhead projector. You may collect the transparencies of other groups. The more transparencies you have, the more easily you see the result.

6. Do you observe any pattern in the points marked? Do you think that the points are marked randomly or they follow some pattern?

Go to the webpage http://www.mathdb.org/, click into Teaching Modules > Fractals, Iterations and Chaos > Lesson 1 > Activity 1D and repeat the game with more ‘transparencies’ and more points on each transparency.

7. Do you think that the points are marked randomly or do they follow some pattern? If you think there is some pattern, is this the same as that for a single transparency with a large number of points?

The pattern that you may observe by placing many transparencies together should be similar to the following:

Figure 1: The Sierpinski Triangle
This is called the Sierpinski Triangle. More properties and another construction method of it will be discussed later. It is one of the most basic fractals, a special kind of geometric figures.

(You can browse the Math Gallery of Mathematical Database (http://www.mathdb.org) for more fractals to have an idea with this kind of figures.)

**Part 2 (Iterations)**

8. When you play the game in part 1, you are continually repeating, using results in the previous step. For each successive step, what exactly is the result that is needed from the previous step?

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9. Iteration is a repetitive process where one step depends on the result of a previous step. Can you think of other examples of iteration, not necessarily in mathematics?

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**Part 3 (Fractals – Self-similarity)**

10. Below shows the Sierpinski Triangle again.

![Figure 2: The Sierpinski Triangle](image)

Let $X$, $Y$ and $Z$ denote the parts of the Sierpinski Triangle inside triangles $AFE$, $BDF$ and $CDE$ respectively. Tick the items that you think are correct.
☐ X is congruent to the large Sierpinski Triangle.
☐ Y is similar to the large Sierpinski Triangle.
☐ X, Y and Z are identical.
☐ The large Sierpinski Triangle is made up of three identical copies of Z.
☐ X is a scaled copy of the large Sierpinski Triangle.

11. X can be divided into three smaller triangles using the same method of dividing the Sierpinski Triangle. How many copies of such smaller triangles are needed to make up the whole large Sierpinski Triangle?

Refer to the Sierpinski Triangle of Figure 2. Each of X, Y and Z is similar to the large one and we say that the Sierpinski Triangle possesses self-similarity. This means that by magnifying a part of the figure, we could get back the original one. In this case, X is a part of the original one and by magnifying it by a linear factor of 2 (i.e. such that each side is doubled), it becomes congruent to the original big Sierpinski Triangle.

Let us now look at an alternative way of constructing the Sierpinski Triangle. Try the following out in the space below.

i. Draw a large equilateral triangle. Shade the whole triangle lightly with a pencil.

ii. Mark the midpoint of the shaded triangle and join the midpoints. Erase the middle inverted triangle and leave the three shaded triangles at the corners.

iii. Repeat Step ii for each remaining shaded triangle and erase the middle inverted triangles.

iv. Repeat Step iii five times.
Here is a sequence of graphs that you should get:

![Figure 3: Alternative construction of the Sierpinski Triangle](image)

12. In the above construction, at each stage we get a figure which consists of many small shaded triangles. How many small shaded triangles are there in Stage 1? Stage 2? Stage 3? Stage $n$?

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**Part 4 (Dimension)**

13. Discuss with your groupmates what you think about the meaning of ‘dimension’.

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14. What are the dimensions of a line, a plane and a cube respectively?

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15. (a) Consider a line segment. Is a line self-similar?

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(b) How many line segments of length 1 are required to form a line segment of length 2?

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16. (a) Now, we consider a square. Is it self-similar? 
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(b) How many squares of side length 1 are required to form a square of side length 2? 
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17. (a) Next, we turn to a cube. Is it self-similar? 
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(b) How many cubes of side length 1 are required to form a cube of side length 2? 
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18. From Questions 15 to 17, doubling the side lengths of a line, a square and a cube require 2, 4 and 8 copies of original figure respectively. Complete the following table.

<table>
<thead>
<tr>
<th>Figure</th>
<th>Number of copies required</th>
<th>Dimension of the figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line segment</td>
<td>_____ = 2()</td>
<td></td>
</tr>
<tr>
<td>Square</td>
<td>_____ = 2()</td>
<td></td>
</tr>
<tr>
<td>Cube</td>
<td>_____ = 2()</td>
<td></td>
</tr>
</tbody>
</table>

What do you observe from the above table? Can you suggest a formula for dimension? 
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**Part 5 (Area of the Sierpinski Triangle)**

We will discuss what a reasonable dimension of the Sierpinski Triangle should be in a moment. Let us first consider the first few steps in the construction of the Sierpinski Triangle by successive removal of the middle inverted triangles, as illustrated in Figure 3.

Let the area of the triangle at Stage 0 be 1. Complete the table on the next page.
<table>
<thead>
<tr>
<th>The ( k )-th stage</th>
<th>A sketch of what you get in the ( k )-th stage</th>
<th>Area that remains</th>
<th>Ratio of ( \frac{A_n}{A_{n-1}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>( A_0 = 1 )</td>
<td>--</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>( A_1 = )</td>
<td>( \frac{A_1}{A_0} )</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>( A_2 = )</td>
<td>( \frac{A_2}{A_1} )</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>( A_3 = )</td>
<td>( \frac{A_3}{A_2} )</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>( A_4 = )</td>
<td>( \frac{A_4}{A_3} )</td>
</tr>
</tbody>
</table>

19. Can you give a general formula for \( A_n \)?

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20. When \( n \) gets larger and larger, what happens to \( A_n \)?

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21. It is commonly believed that a figure of (2-dimensional) area 0 should not be of dimension 2. In view of this, should the dimension of the Sierpinski Triangle be 2? If not, do you think the Sierpinski Triangle is of dimension 1?

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\textbf{Part 6 (Fractional Dimension of the Sierpinski Triangle)}

22. As discussed in Part 3, how many copies of Sierpinski Triangle are required to produce another triangle with double side lengths?

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23. With the result in the discussion in Part 4, what do you think the dimension of the Sierpinski Triangle should be?

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24. Is your answer in Question 23 smaller than 2? Does this match with the result obtained in Part 5?

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The non-integral dimension of the Sierpinski Triangle is another interesting property possessed by fractals besides self-similarity.

\textbf{Part 7 (Hitting the Sierpinski Triangle)}

Suppose we have built a Sierpinski Triangle using elastic strings. When we hit the Sierpinski Triangle, we can imagine that we will produce a wave which travels away from the point where we hit the Sierpinski Triangle. What do you think the wave will look like?
Go to the webpage [http://www.mathdb.org/](http://www.mathdb.org/), click into Teaching Modules > Fractals, Iterations and Chaos > Lesson 1 > Activity 7, and you will find out how the wave looks.

The interesting point here is that mathematicians found that small amounts of wave energy can propagate at arbitrarily large speed. In other words, the speed of this traveling wave increases if we pass to smaller and smaller scales. In particular, there is no maximum propagation speed for waves on the Sierpinski Triangle. In fact mathematicians think that it is very likely that waves on the Sierpinski Triangle propagate with an infinite speed. If that is the case it will be a striking contrast to waves in a one-dimensional elastic string, which we all believe to propagate with finite speeds.

The above-mentioned is actually an active area of current research. In fact the theoretical study of fractals only began in the 1970’s, although mathematicians knew some examples of fractals in the 1900’s already. We will meet some other current research interests in the lessons to come. We hope these will give you some idea about mathematical researches.

**Summary**

1. The Random Walk Game gives a figure that looks like the Sierpinski Triangle.
2. The Sierpinski Triangle is an example of fractals. It is self-similar and has a dimension of \( \frac{\log 3}{\log 2} \). Self-similarity and fractional dimension are two properties possessed by many fractals.
3. The Sierpinski Triangle also shows interesting properties when a wave is applied to it.

~ End of worksheet ~

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1 The animations are adopted from [http://www.mathlab.cornell.edu/~gibbons/heat.htm](http://www.mathlab.cornell.edu/~gibbons/heat.htm). We thank Michael Gibbons and Arjun Raj for allowing us to reproduce them here.