CHAPTER 1

Lebesgue Measure

We know that the length of an interval is defined to be the difference between two end points. In this chapter, we would like to extend the idea of "length" to arbitrary (or at least, as many as possible) subsets of $\mathbb{R}$. To begin with, let's recall two important results in topology.

1. Review

PROPOSITION 1.1. *Every open subset V of $\mathbb{R}$ is a countable union of disjoint open intervals.*

PROOF. For each $x \in V$, there is an open interval I_x with rational endpoints such that $x \in I_x \subseteq V$. Then the collection $\{I_x\}_{x\in V}$ is evidently countable and

$$V = \bigcup_{x\in V} I_x.$$

Next, we prove it is always possible to have a disjoint collection. Since $\{I_x\}_{x\in V}$ is a countable collection, we can enumerate the open intervals as $I_1 = (a_1, b_1), I_2 = (a_2, b_2), \ldots$. For each $n \in \mathbb{N}$, define

$$\alpha_n = \inf\{\, x \in \mathbb{R} \colon x \le a_n \text{ and } (x, b_n) \subseteq V \,\}$$

and

$$\beta_n = \sup\{\, x \in \mathbb{R} \colon x \ge b_n \text{ and } (a_n, x) \subseteq V \,\}.$$

Then $\{\, (\alpha_n, \beta_n) \,\}_{n\in\mathbb{N}}$ is a disjoint collection of open intervals with union V. □

THEOREM 1.2 (Lindelöf's Theorem). *Let C be a collection of open subsets of $\mathbb{R}$. Then there is a countable sub-collection $\{O_i\}_{i\in\mathbb{N}}$ of C such that*

$$\bigcup_{O\in C} O = \bigcup_{i=1}^{\infty} O_i.$$

PROOF. Let $U = \bigcup_{O\in C} O$. For any $x \in U$ there is $O \in C$ with $x \in O$. Take an open interval I_x with rational endpoints such that $x \in I_x \subseteq O$. Then $U = \bigcup_{x\in U} I_x$ is a countable union of open intervals. Replace I_x by the set $O \in C$ which contains it, the result follows. □

2. Lebesgue outer measure

As in the Archimedean idea of finding area of a circle (approximated polygons), we define the *Lebesgue outer measure* $m^*\colon \mathcal{P}(\mathbb{R}) \to [0,\infty]$ by

$$m^*(A) = \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k)\colon A \subseteq \bigcup_{k=1}^{\infty} I_k \text{ and each } I_k \text{ being open interval in } \mathbb{R} \right\}.$$

REMARK. By Lindelöf's Theorem, the countability of the covering is not important here.

Here are some basic properties of Lebesgue outer measure, all of them can be proved easily by the definition of m^*.

(i) $m^*(A) = 0$ if A is at most countable.

(ii) m^* is *monotonic*, i.e. $m^*(A) \leq m^*(B)$ whenever $A \subseteq B$.

(iii) $m^*(A) = \inf\{\, m^*(O)\colon A \subseteq O \text{ and } O \text{ is open}\,\}$. (Hint: it sufficies to prove $m^*(A) \geq$ R.H.S., which is equivalent to $m^*(A) + \varepsilon >$ R.H.S. for any $\varepsilon > 0$.)

(iv) $m^*(A + x) = m^*(A)$ for all $x \in \mathbb{R}$. (Translation-invariant)

(v) $m^*(\bigcup_{k\in\mathbb{N}} A_k) \leq \sum_{k=1}^{\infty} m^*(A_k)$. (Countable subadditivity)

(vi) If $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$ and $m^*(B \setminus A) = m^*(B)$ for all $B \subseteq \mathbb{R}$.

(vii) If $m^*(A \triangle B) = 0$, then $m^*(A) = m^*(B)$.

REMARK. In (v), even if A_k's are disjoint, the equality may not hold.

THEOREM 1.3. *For any interval $I \subseteq \mathbb{R}$, $m^*(I) = \ell(I)$.*

PROOF. We first assume $I = [a, b]$ is a closed and bounded interval. Consider the countable open interval cover $\{(a - \varepsilon, b + \varepsilon)\}$ of I, we have $m^*(I) \leq b - a + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $m^*(I) \leq b - a$.

To get the opposite result, we need show for any $\varepsilon > 0$, $m^*(I) + \varepsilon \geq b - a$. Note that there is a countable open interval cover $\{I_k\}_{k\in\mathbb{N}}$ of I satisfying

$$m^*(I) + \varepsilon > \sum \ell(I_k).$$

By Heine-Borel Theorem, there is a finite subcover $\{I_{n_k}\}$ of $\{I_k\}$. Then

$$\sum \ell(I_{n_k}) > b - a \qquad \text{(why?)}$$

and it follows that

$$m^*(I) + \varepsilon > \sum \ell(I_k) \geq \sum \ell(I_{n_k}) > b - a.$$

Letting $\varepsilon \to 0$, $m^*(I) \geq b - a$. Hence, $m^*(I) = b - a$.

Next, we consider the case where $I = (a, b), [a, b)$, or $(a, b]$ which is bounded but not closed. Clearly, $m^*(I) \leq m^*(\overline{I}) = b - a$. On the other hand, if $\varepsilon > 0$ is sufficiently small then there is a closed and bounded interval $I' = [a + \varepsilon, b - \varepsilon] \subseteq I$. By monotonicity, $m^*(I) \geq m^*(I') = b - a - 2\varepsilon$. Letting $\varepsilon \to 0$ gives $m^*(I) \geq b - a$. Hence, $m^*(I) = b - a$.

Finally, if I is unbounded then the result is trivial since in that case I contains interval of arbitrarily large length. □

3. Non-measurability

Theorem 1.4. *Let $\mathfrak{M} \subseteq \mathcal{P}(\mathbb{R})$ be a translation-invariant σ-algebra containing all intervals, and $m\colon \mathfrak{M} \to [0, \infty]$ be a translation-invariant, countably additive measure such that*

$$m(I) = \ell(I) \qquad \text{for all interval } I.$$

Then there exists a set $S \notin \mathfrak{M}$.

Proof. Define an equivalent relation $x \sim y$ if and only if $x - y$ is rational. Then $\mathbb{R}$ is partitioned into disjoint *cosets* $[x] = \{\, y \in \mathbb{R} \colon x \sim y \,\}$.

By Axiom of Choice and Archimedean property of $\mathbb{R}$, there exists $S \subseteq [0, 1]$ such that the intersection of S with each coset contains exactly one point.

Enumerate $\mathbb{Q} \cap [-1, 1]$ into $r_1, r_2, \ldots$. Then the sets $S + r_i$ are disjoint and

$$[0, 1] \subseteq \bigcup_{i \in \mathbb{N}} (S + r_i) \subseteq [-1, 2].$$

If $S \in \mathfrak{M}$, then by monotonicity and countable additivity of m we have

$$1 \leq \sum_{i \in \mathbb{N}} m(S + r_i) \leq 3,$$

which is impossible since $m(S + r_i) = m(S)$ for all $i \in \mathbb{N}$. □

4. Measurable sets and Lebesgue measure

As it is mentioned before, the outer measure does not have countable additivity. One may try to restrict the outer measure m^* to a σ-algebra $\mathfrak{M} \subsetneq \mathcal{P}(\mathbb{R})$ such that the new measure has all the properties we wanted.

Definition (Measurability). A set $E \subseteq \mathbb{R}$ is said to be *measurable* if, for all $A \subseteq \mathbb{R}$, one has

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c). \tag{1}$$

Since m^* is known to be subadditive, (1) is equivalent to

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

The family of all measurable sets is denoted by $\mathfrak{M}$. We will see later $\mathfrak{M}$ is a σ-algebra and translation-invariant containing all intervals. The set function $m \colon \mathfrak{M} \to [0, \infty]$ defined by

$$m(E) = m^*(E) \qquad \text{for all } E \in \mathfrak{M}$$

is called *Lebesgue measure.*

Observe that

- $E \in \mathfrak{M} \Leftrightarrow E^c \in \mathfrak{M}$.
- $\phi \in \mathfrak{M}$ and $\mathbb{R} \in \mathfrak{M}$ because $m^*(A) = m^*(A \cap \phi) + m^*(A \cap \mathbb{R})$ for all $A \subseteq \mathbb{R}$.
- $m^*(E) = 0 \Rightarrow E \in \mathfrak{M}$ because $m^*(A \cap E) + m^*(A \cap E^c) = m^*(A \cap E^c) \leq m^*(A)$ for all $A \subseteq \mathbb{R}$.

PROPOSITION 1.5. *If $E_1, E_2 \in \mathfrak{M}$ then $E_1 \cup E_2 \in \mathfrak{M}$. (Therefore, $\mathfrak{M}$ is an algebra.)*

PROOF. For all $A \subseteq \mathbb{R}$ one has

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c) \qquad (\because E_1 \in \mathfrak{M}) \\ &= m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c) \qquad (\because E_2 \in \mathfrak{M}) \\ &\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \end{aligned}$$

because m^* is subadditive and

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_1^c \cap E_2).$$

□

REMARK. Proposition 1.5 can be easily extended to a finite union of measurable sets, in fact it can be extended to a countable union. In order to do so, we need the following result.

LEMMA 1.6. *Let $E_1, E_2, \ldots, E_n$ be disjoint measurable sets. Then for all $A \subseteq \mathbb{R}$, we have*

$$m^*\left(A \cap \left[\bigcup_{i=1}^n E_i\right]\right) = \sum_{i=1}^n m^*(A \cap E_i).$$

PROOF. Since $E_n \in \mathfrak{M}$, we have

$$\begin{aligned} m^*\left(A \cap \left[\bigcup_{i=1}^n E_i\right]\right) &= m^*\left(A \cap \left[\bigcup_{i=1}^n E_i\right] \cap E_n\right) + m^*\left(A \cap \left[\bigcup_{i=1}^n E_i\right] \cap E_n^c\right) \\ &= m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_{i=1}^{n-1} E_i\right]\right) \end{aligned}$$

Repeat the process again and again, until we get

$$m^*\left(A\cap\left[\bigcup_{i=1}^n E_i\right]\right)=\sum_{i=1}^n m^*(A\cap E_i).$$

□

REMARK. In fact, if $\{E_i\}_{i\in\mathbb{N}}$ is a sequence of disjoint measurable sets, then

$$m^*\left(A\cap\left[\bigcup_{i=1}^\infty E_i\right]\right)=\sum_{i=1}^\infty m^*(A\cap E_i).$$

This is because for all $n\in\mathbb{N}$ one has

$$\begin{aligned}m^*\left(A\cap\left[\bigcup_{i=1}^\infty E_i\right]\right)&\geq m^*\left(A\cap\left[\bigcup_{i=1}^n E_i\right]\right)\\&=\sum_{i=1}^n m^*(A\cap E_i).\end{aligned}$$

Letting $n\to\infty$ lead to

$$m^*\left(A\cap\left[\bigcup_{i=1}^\infty E_i\right]\right)\geq\sum_{i=1}^\infty m^*(A\cap E_i).$$

The opposite inequality follows from countable subadditivity.

THEOREM 1.7. *Let $\{E_i\}_{n\in\mathbb{N}}$ be a sequence of measurable sets, then $E=\bigcup_{i=1}^\infty E_i$ is also measurable. Moreover, if $E_1,E_2,\ldots$ are disjoint then $m(E)=\sum_{i=1}^\infty m(E_i)$.*

This is called the *countable additivity* which can be proved by putting $A=\mathbb{R}$ in the remark of Lemma 1.6.

PROOF. We first assume E_1, E_2, …are disjoint. Then for all $A\subseteq\mathbb{R}$, $n\in\mathbb{N}$ we have

$$\begin{aligned}m^*(A)&=m^*\left(A\cap\bigcup_{i=1}^n E_i\right)+m^*\left(A\cap\left(\bigcup_{i=1}^n E_i\right)^c\right)\\&\geq\sum_{i=1}^n m^*(A\cap E_i)+m^*(A\cap E^c).\end{aligned}$$

Letting $n\to\infty$,

$$\begin{aligned}m^*(A)&\geq\sum_{i=1}^\infty m^*(A\cap E_i)+m^*(A\cap E^c)\\&=m^*(A\cap E)+m^*(A\cap E^c).\end{aligned}$$

This proved E is measurable.

Now, if E_1, E_2, …are not disjoint, we let

$$F_1=E_1,\qquad F_2=E_2\setminus F_1,\qquad F_3=E_3\setminus(F_1\cup F_2),$$

and in general $F_k = E_k \setminus \bigcup_{i=1}^{k-1} F_i$ for $k \geq 2$. Then F_1, F_2, ... are disjoint and $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$. Since $\mathfrak{M}$ is an algebra, F_1, F_2, ... are all measurable. So $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$ is measurable. □

Remark. Now, $\mathfrak{M}$ is proved to be a σ-algebra. The next result shows that all Borel sets are measurable. Recall that the family of Borel sets in $\mathbb{R}$ is, by definition, the smallest σ-algebra containing all open subsets of $\mathbb{R}$.

Theorem 1.8. *$\mathfrak{M}$ contains all Borel subsets of $\mathbb{R}$.*

Proof. It suffices to show that $(a, \infty) \in \mathfrak{M}$ for all $a \in \mathbb{R}$ (why?). Let $A \in \mathbb{R}$. We need to show that

$$m^*(A) \geq m^*(A \cap (-\infty, a]) + m^*(A \cap (a, \infty)).$$

Without loss of generality, we may assume $m^*(A) < \infty$. For convenience, let $A_1 = A \cap (-\infty, a]$ and $A_2 = A \cap (a, \infty)$. Then we need to show

$$m^*(A) + \varepsilon \geq m^*(A_1) + m^*(A_2) \qquad \text{for all } \varepsilon > 0.$$

By the definition of $m^*(A)$, there is a countable open interval cover $\{I_n\}_{n \in \mathbb{N}}$ of A with

$$m^*(A) + \varepsilon > \sum_{n=1}^{\infty} \ell(I_n).$$

Let $I_n' = I_n \cap (-\infty, a]$ and $I_n'' = I_n \cap (a, \infty)$, then $\{I_n'\}$, $\{I_n''\}$ are, respectively, interval covers of A_1 and A_2 (note that they may not be open interval covers). Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \ell(I_n) &= \sum_{n=1}^{\infty} \ell(I_n') + \sum_{n=1}^{\infty} \ell(I_n'') \\
&= \sum_{n=1}^{\infty} m^*(I_n') + \sum_{n=1}^{\infty} m^*(I_n'') \qquad (\because m^* = \ell \text{ for intervals}) \\
&\geq m^*\left(\bigcup_{n=1}^{\infty} I_n'\right) + m^*\left(\bigcup_{n=1}^{\infty} I_n''\right) \qquad (\because \text{countable subadditivity}) \\
&\geq m^*(A_1) + m^*(A_2) \qquad (\because \text{monotonicity})
\end{aligned}$$

So, $m^*(A) + \varepsilon > \sum_{n=1}^{\infty} \ell(I_n) \geq m^*(A_1) + m^*(A_2)$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, $m^*(A) \geq m^*(A_1) + m^*(A_2)$. This proved that $(a, \infty) \in \mathfrak{M}$. □

Remark. Since $\mathfrak{M}$ is a σ-algebra, $(-\infty, a] \in \mathfrak{M}$ and $(-\infty, a) = \bigcup_{n=1}^{\infty}(-\infty, a - 1/n] \in \mathfrak{M}$. It follows that $(a, b) \in \mathfrak{M}$ since $(a, b) = (-\infty, b) \cap (a, \infty)$. As $\mathfrak{M}$ is a σ-algebra containing all open intervals, it must contain all open sets (recall that every open set is countable union of open intervals by Proposition 1.1). Therefore, $\mathfrak{M}$ contains all Borel sets.

Proposition 1.9. *$\mathfrak{M}$ is translation invariant: for all $x \in \mathbb{R}$, $E \in \mathfrak{M}$ implies $E + x \in \mathfrak{M}$.*

PROOF. For all $A \in \mathbb{R}$, we have

$$\begin{aligned} m^*(A) &= m^*(A - x) \\ &= m^*\left((A - x) \cap E\right) + m^*\left((A - x) \cap E^c\right) \\ &= m^*\left(((A - x) \cap E) + x\right) + m^*\left(((A - x) \cap E^c) + x\right) \\ &= m^*\left(A \cap (E + x)\right) + m^*\left(A \cap (E + x)^c\right) \end{aligned}$$

□

THEOREM 1.10 (Littlewood's 1st Principle). *Every measurable set of finite measure is nearly a finite union of disjoint open intervals, in the sense*

- *If E is measurable and $m(E) < \infty$, then for any $\varepsilon > 0$ there is a finite union U of open intervals such that $m^*(U \triangle E) < \varepsilon$. (Clearly, the intervals can be chosen to be disjoint.)*
- *If for any $\varepsilon > 0$ there is a finite union U of open intervals such that $m^*(U \triangle E) < \varepsilon$, then E is measurable. (The finiteness assumption $m^*(E) < \infty$ is not essential.)*

REMARK. Let $E \in \mathbb{R}$ be given. Then the following statements are equivalent.

(1) E is measurable.

(2) For any $\varepsilon > 0$, there is an open set $O \supseteq E$ such that $m^*(O \setminus E) < \varepsilon$.

(3) For any $\varepsilon > 0$, there is a closed set $F \subseteq E$ such that $m^*(E \setminus F) < \varepsilon$.

(4) There is a $G \in G_\delta$ such that $E \subseteq G$ and $m^*(G \setminus E) = 0$.

(5) There is a $F \in F_\sigma$ such that $E \supseteq F$ and $m^*(E \setminus F) = 0$.

Assume $m^*(E) < \infty$, the above statements are equivalent to

(6) For any $\varepsilon > 0$, there is a finite union U of open intervals such that $m^*(U \triangle E) < \varepsilon$.

PROOF. If we can prove (1), (2), and (4) are equivalent, then it is easy to see that (2) and (3) are equivalent, because one implies another by replacing E with E^c. Similarly, (4) and (5) are equivalent.

To show (1) $\Rightarrow$ (2)

We first consider a simple case $m(E) < \infty$. For any $\varepsilon > 0$, there is a countable open interval cover $\{I_n\}$ of E such that $\sum_{n=1}^{\infty} \ell(I_n) < m(E) + \varepsilon$. Take $O = \bigcup_{n=1}^{\infty} I_n$, we see that O is open and $O \supseteq E$. Also, we have

$$m(O \setminus E) = m(O) - m(E) \leq \sum_{n=1}^{\infty} m(I_n) - m(E) < \varepsilon.$$

Here we use the assumption $m(E) < \infty$ and the countable subadditivity of m.

For the case $m(E) = \infty$, we write $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n = E \cap [-n, n]$. This is a countable union of measurable sets of finite measure. By the above result there is an open set O_n such that $O_n \supseteq E_n$ and $m^*(O_n \setminus E_n) < \frac{\varepsilon}{2^n}$. Take $O = \bigcup_{n=1}^{\infty} O_n$, then O is open and $O \supseteq E$. It remains to show $m(O \setminus E) < \varepsilon$.

Note that $O \setminus E \subseteq \bigcup_{n=1}^{\infty} O_n \setminus E_n$, by countable subadditivity of m we have

$$m(O \setminus E) \leq \sum_{n=1}^{\infty} m(O_n \setminus E_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Hence, we have proved that (1) $\Rightarrow$ (2).

<u>To show (2) $\Rightarrow$ (4)</u>

For any $n \in \mathbb{N}$, let O_n be an open set such that $O_n \supseteq E$ and $m^*(O_n \setminus E) < 1/n$. Take $G = \bigcap_{n=1}^{\infty} O_n \in G_\delta$, then

$$m^*(G \setminus E) \leq m^*(O_n \setminus E) < \frac{1}{n}.$$

Letting $n \to \infty$, the result follows.

<u>To show (4) $\Rightarrow$ (1)</u>

The existence of G guarantees $E = G \setminus (G \setminus E)$ is measurable since both G and $G \setminus E$ are measurable (G is Borel set and $G \setminus E$ is of measure zero).

Hence, (1), (2), (3), (4), (5) are equivalent.

<u>To show (2) $\Rightarrow$ (6)</u> (with finiteness assumption $m^*(E) < \infty$)

Let $\varepsilon > 0$ be given. Let O be an open set such that $O \supseteq E$ and $m(O \setminus E) < \varepsilon/2$. Write $O = \bigcup_{n=1}^{\infty} I_n$ to be a countable union of disjoint open intervals. By the countable additivity of m, $m(O) = \sum_{n=1}^{\infty} \ell(I_n)$. Let k be a positive integer such that $\sum_{n=1}^{k} \ell(I_n) > m(O) - \varepsilon/2$. (The finiteness assumption has been used here to guarantee that $m(O) < \infty$.)

Take $U = \bigcup_{n=1}^{k} I_n$. Note that $m(O \setminus U) = m(O) - m(U) < \varepsilon/2$, so

$$\begin{aligned} m(U \triangle E) &= m(U \setminus E) + m(E \setminus U) \\ &\leq m(O \setminus E) + m(O \setminus U) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

The finiteness assumption is essential here. The above result is false if we allow E to have infinite measure. A counter example is $E = \bigcup_{n=1}^{\infty} (2n, 2n+1)$.

To show (6) $\Rightarrow$ (2) (without finiteness assumption $m^*(E) < \infty$)

Let $\varepsilon > 0$ be given and U be a finite union of open intervals described in (6). Then $m^*(E \setminus U) < \varepsilon$, we take an open set $O' \supseteq E \setminus U$ such that $m^*(O') < \varepsilon$ (how to do this?). Then $O = U \cup O'$ is an open set containing E with $m^*(O \setminus E) \leq m^*(U \setminus E) + m^*(O') < 2\varepsilon$. $\square$

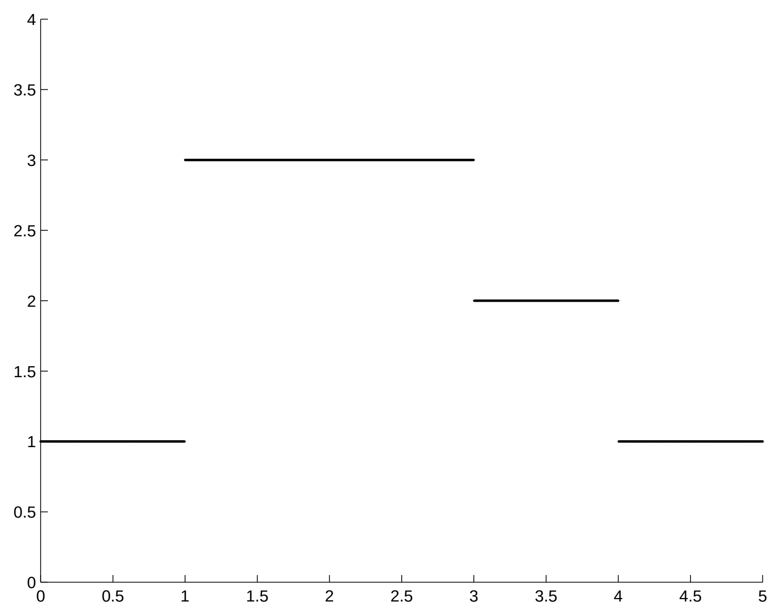
EXERCISE 1.1. Let $A \in \mathbb{R}$, prove that there is a measurable set $B \supseteq A$ with $m^*(A) = m^*(B)$.

5. Step functions and simple functions

DEFINITION. A function $\psi\colon [a,b] \to \mathbb{R}$ is called *step function* if

$$\psi(x) = c_i \qquad (x_{i-1} < x < x_i)$$

for some partition $\{x_0, x_1, \dots, x_n\}$ of $[a,b]$ and some constants $c_1, c_2, \dots, c_n$.



An example of step function

LEMMA 1.11. *Let ψ_1, ψ_2 be step functions on $[a,b]$. Then $\psi_1 \pm \psi_2$, $\alpha\psi_1 + \beta\psi_2$, $\psi_1\psi_2$, $\psi_1 \wedge \psi_2$, and $\psi_1 \vee \psi_2$ are all step functions, where $\alpha, \beta \in \mathbb{R}$. Also, if $\psi_2 \neq 0$ on $[a,b]$, then ψ_1/ψ_2 is also step function.*

REMARK. $(f \wedge g)(x) = \min\{f(x), g(x)\}$ and $(f \vee g)(x) = \max\{f(x), g(x)\}$.

LEMMA 1.12. *Let ψ be a step function on $[a,b]$ and let $\varepsilon > 0$. Then there is a continuous function g on $[a,b]$ such that $\psi = g$ on $[a,b]$ except on a set of measure less than ε, i.e.*

$$m\left(\{\, x \in [a,b] \colon \psi(x) \neq g(x) \,\}\right) < \varepsilon.$$

Proof. Easy! One can find a piecewise linear function g with the stated property. $\square$

Definition. Let $E \in \mathfrak{M}$. A function $f\colon E \to \mathbb{R}$ is called a *simple function* if there exists $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $E_1, E_2, \ldots, E_n \in \mathfrak{M}$ such that

$$f = \sum_{i=1}^{k} a_i \mathcal{X}_{E_i} \tag{2}$$

Remark. Step function is simple, $\mathcal{X}_{\mathbb{Q}}$ is simple but not step function.

Proposition 1.13. *Let $f\colon [a,b] \to \mathbb{R}$ be a simple function. For any $\varepsilon > 0$, there is a step function $\psi\colon [a,b] \to \mathbb{R}$ such that $f = \psi$ except on a set of measure less than ε.*

Proof. Let f be given by (2), we may assume $E_1, E_2, \ldots, E_n \subseteq E$. By Littlewood's 1st Principle, there is a finite union of disjoint open intervals U_i such that $m(U_i \triangle E_i) < \varepsilon/n$. Then

$$f = \sum_{i=1}^{n} a_i \mathcal{X}_{U_i} \qquad \text{except on } A = \bigcup_{i=1}^{n} (U_i \triangle E_i)\,,$$

where $m(A) < \sum_{i=1}^{n} \varepsilon/n = \varepsilon$. $\square$

Remark. By Lemma 1.12, one can find a continuous function with the same property. Moreover, if f satisfies $m \le f \le M$ on $[a,b]$ then ψ can be chosen such that $m \le \psi \le M$ (reason: replace ψ by $(m \vee \psi) \wedge M$ if necessary).

6. Measurable functions

Definition. A function $f\colon E \to [-\infty, \infty]$ is said to be *measurable* (or *measurable on* E) if $E \in \mathfrak{M}$ and

$$f^{-1}((a,\infty]) \in \mathfrak{M}$$

for all $a \in \mathbb{R}$.

In fact, there is a more general definition for measurability which we will not use here. The definition goes as follows.

Definition. Let X be a measurable space[1] and Y be a topological space. A function $f\colon X \to Y$ is called *measurable* if $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

Remark. Simple functions, step functions, continuous functions, and monotonic functions are measurable.

Proposition 1.14. *Let $E \in \mathfrak{M}$ and $f\colon E \to [-\infty, \infty]$. Then the following 4 statements are equivalent:*

- *$f^{-1}((a,\infty]) \in \mathfrak{M}$ for all $a \in \mathbb{R}$.*

[1]A measurable space, by definition, is a nonempty set with an associated σ-algebra on it.

- $f^{-1}([a,\infty]) \in \mathfrak{M}$ *for all* $a \in \mathbb{R}$.
- $f^{-1}([-\infty,a)) \in \mathfrak{M}$ *for all* $a \in \mathbb{R}$.
- $f^{-1}([-\infty,a]) \in \mathfrak{M}$ *for all* $a \in \mathbb{R}$.

REMARK. The above statements imply $f^{-1}(a) \in \mathfrak{M}$ for all $a \in [-\infty,\infty]$. The converse is not true.

PROOF. The first one is clearly equivalent to the fourth one since $f^{-1}((a,\infty]) = E \setminus f^{-1}([-\infty,a])$. Similarly, the second and the third statements are equivalent. It remains to show the first two statements are equivalent, but this follows immediately from

$$f^{-1}([a,\infty]) = \bigcap_{n=1}^{\infty} f^{-1}\left((a-\frac{1}{n},\infty]\right) \qquad \text{and} \qquad f^{-1}((a,\infty]) = \bigcup_{n=1}^{\infty} f^{-1}\left([a+\frac{1}{n},\infty]\right).$$

□

PROPOSITION 1.15. *Let* $E \in \mathfrak{M}$, $f\colon E \to [-\infty,\infty]$ *and* $g\colon E \to [-\infty,\infty]$. *If* $f = g$ *almost everywhere*[2] *on* E *then the measurability of* f *and* g *are the same.*

PROOF. Simply note that

$$m^*\left(\{\, x \in E\colon f(x) > a \,\}\triangle\{\, x \in E\colon g(x) > a \,\}\right) \leq m^*\left(\{\, x \in E\colon f(x) \neq g(x) \,\}\right) = 0.$$

This implies the measurability of the sets $\{\, x \in E\colon f(x) > a \,\}$ and $\{\, x \in E\colon g(x) > a \,\}$ are the same. □

PROPOSITION 1.16. *Let* f, g *be measurable extend real-valued functions on* $E \in \mathfrak{M}$. *Then the following functions are all measurable on* E:

$$f + c, cf, f \pm g, fg$$

where $c \in \mathbb{R}$.

REMARK. One may worry that cf, $f \pm g$, fg may not be defined at some points (for example, if $f = \infty$ and $g = -\infty$ then $f + g$ is meaningless). There are two ways to deal with this problem.

(1) adopt the convention $0 \cdot \infty = 0$.
(2) assume f, g are finite almost everywhere or cf, $f \pm g$, fg are meaningful almost everywhere.

PROOF. We only prove $f + g$ and fg are measurable, since the others are easy or similar.

[2]We say that a property P holds *almost everywhere* on E if the set $\{\, x \in E\colon P \text{ fails to hold at } x \,\}$ has measure zero.

To prove $f+g$ is measurable, one should consider the set

$$\begin{aligned} E_a &= \{\, x \in E \colon f(x)+g(x) > a \,\} \\ &= \{\, x \in E \colon f(x) > a - g(x) \,\} \\ &= \bigcup_{r\in\mathbb{Q}} \{\, x \in E \colon f(x) > r > a - g(x) \,\} \\ &= \bigcup_{r\in\mathbb{Q}} \{\, x \in E \colon f(x) > r \,\} \cap \{\, x \in E \colon r > a - g(x) \,\} \end{aligned}$$

If $f(x) = \infty$ or $g(x) = \infty$ then $x \in E_a$ by convention. Now $E_a \in \mathfrak{M}$ because E_a is countable union of measurable sets.

Next, we prove f^2 is measurable. For $a \geq 0$,

$$\{\, x \in E \colon f^2(x) > a \,\} = \{\, x \in E \colon f(x) > \sqrt{a} \,\} \cup \{\, x \in E \colon f(x) < -\sqrt{a} \,\}$$

is measurable. For $a < 0$, $\{\, x \in E \colon f^2(x) > a \,\} = E$ is also measurable. Therefore, f^2 is measurable and it is valid even if f takes values $\pm\infty$.

So, if f and g are assumed to be finite, then

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$$

is measurable on E. □

Exercise 1.2. Find two measurable functions f, g from $\mathbb{R}$ to $\mathbb{R}$ such that $f \circ g$ is not measurable.

Proposition 1.17. *Let $\{f_n\}_{n\in\mathbb{N}}$ be measurable extended real-valued functions on a measurable set E. Then*

$$f_1 \vee f_2 \cdots \vee f_n, \qquad \sup_{n\in\mathbb{N}} f_n, \qquad \varlimsup_{n\to\infty} f_n$$

are all measurable on E. Similar results hold if $\vee$, $\sup$ and $\varlimsup$ are replaced by $\wedge$, $\inf$, and $\varliminf$.

Proof. Simply note that

$$\begin{aligned} (f_1 \vee f_2 \cdots \vee f_n)^{-1}((a,\infty)) &= \bigcup_{k=1}^{n} f_k^{-1}((a,\infty)) \\ \left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}((a,\infty)) &= \bigcup_{k=1}^{\infty} f_k^{-1}((a,\infty)) \\ \varlimsup_{n\to\infty} f_n &= \inf_{N\in\mathbb{N}}\left(\sup_{k\geq N} f_k\right) \end{aligned}$$

□

THEOREM 1.18 (Littlewood's 2nd Principle). *Let $E \in \mathfrak{M}$ with $m(E) < \infty$, $f: E \to [-\infty, \infty]$ be measurable and finite almost everywhere. For any $\varepsilon > 0$, there is a simple function ϕ such that*

$$|f - \phi| < \varepsilon \text{ on } E \text{ except on a set of measure less than } \varepsilon.$$

REMARK. If $E = [a, b]$ is closed and bounded interval, we can find a step function g and a continuous function h play the role of ϕ. This is because simple function can be approximated by step function and step function can be approximated by continuous function.

If f satisfies an additional condition $m \leq f \leq M$, then ϕ, g, and h can be chosen to be bounded below by m and above by M.

The condition $m(E) < \infty$ in Littlewood's 2nd Principle is essential. You can see if this condition is dropped then taking $f(x) = x$ will give a counter example.

To prove Littlewood's 2nd Principle, we introduce a lemma.

LEMMA 1.19. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{R}$ (or any measure space[3]) such that*

$$F_1 \supseteq F_2 \supseteq \cdots .$$

Denote $F_\infty = \bigcap_{n\in\mathbb{N}} F_n$. If $m(F_1) < \infty$ then

$$m(F_\infty) = \lim_{n\to\infty} m(F_n).$$

PROOF. Write $F_1 = F_\infty \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_3) \cup \cdots$ as disjoint union and use the countable additivity of m. □

REMARK. Lemma 1.19 is false if the condition $m(F_1) < \infty$ is missing. To see this, we take $F_n = (n, \infty)$.

Now, we are ready to prove Littlewood's 2nd Principle.

PROOF. Let $\varepsilon > 0$ be given. Our proof is divided into two steps.

<u>Step I</u> Assume $m \leq f \leq M$ for some $m, M \in \mathbb{R}$.

We divide $[m, M]$ into n subintervals such that the length of each subinterval is less than ε. Symbolically, we take the partition points as follows:

$$m = y_0 < y_1 < \cdots < y_n = M \qquad \text{with } y_i - y_{i-1} < \varepsilon \text{ for } 1 \leq i \leq n.$$

Let $E_1 = \{ x \in E : m \leq f(x) \leq y_1 \}$, $E_2 = \{ x \in E : y_1 < f(x) \leq y_2 \}$, ..., $E_n = \{ x \in E : y_{n-1} < f(x) \leq M \}$. Now, take $\phi = y_1\mathcal{X}_{E_1} + y_2\mathcal{X}_{E_2} + \cdots + y_n\mathcal{X}_{E_n}$. Since $E_1, E_2, \ldots, E_n$ are all measurable (why?), ϕ is simple and satisfies the inequality $|f - \phi| < \varepsilon$ with no exceptions.

[3]A measure space is a nonempty set associated with a σ-algebra $\mathfrak{M}$ to it and a measure μ on $\mathfrak{M}$.

Step II General case.

We let

$$F_n = \{ x \in E \colon |f(x)| \ge n \}.$$

Then $F_1 \supseteq F_2 \supseteq \cdots$. Note that $m(F_1) \le m(E) < \infty$ and $m(F_\infty) = 0$ by assumption, apply Lemma 1.19 there exists $N \in \mathbb{N}$ such that

$$m(F_N) < \varepsilon.$$

Now, let $f^* = (-N \vee f) \wedge N$, then $f = f^*$ on E except on a set of measure less than ε. From the result of Step I, there is a simple function ϕ such that $|f^* - \phi| < \varepsilon$ on E. Hence

$$|f - \phi| < \varepsilon \text{ on } E \text{ except on a set of measure less than } \varepsilon.$$

□

Corollary 1.19.1. *There is a sequence of simple functions ϕ_n such that $\phi_n \to f$ pointwisely almost everywhere on E. If $E = [a, b]$, there are also sequence of step functions and sequence of continuous functions converging to f pointwisely almost everywhere on $[a, b]$.*

Proof. Applying Littlewood's 2nd Principle to $\varepsilon = 1/2^n$, there are simple functions ϕ_n and sets A_n with $m(A_n) < 1/2^n$ such that

$$|f - \phi_n| < \frac{1}{2^n} \qquad \text{on } E \setminus A_n.$$

Let $A = \overline{\lim} A_n := \bigcap_{k=1}^{\infty} (\bigcup_{n=k}^{\infty} A_n)$, then $m(A) = 0$ (why?). The proof is completed by noting that $\phi_n \to f$ pointwisely on $E \setminus A$. □

Remark. In fact, the sequence ϕ_n can be chosen so that $\phi_n \to f$ pointwisely everywhere on E. For example, we can first divide the interval [-n,n] into $2n^2$ subintervals such that each subinterval has length $1/n$, i.e. choose

$$-n = y_0 < y_1 < \cdots < y_{2n^2} = n$$

such that $y_i - y_{i-1} = 1/n$ for all i. Then let

$$\phi_n(x) = \begin{cases} y_i & \text{if } y_i \le f(x) < y_{i+1} \text{ for some } i \\ n & \text{if } f(x) \ge n \\ -n & \text{if } f(x) < -n \end{cases}$$

Theorem 1.20 (Littlewood's 3rd Principle / Egoroff's Theorem). *Let $E \in \mathfrak{M}$ with $m(E) < \infty$, $f \colon E \to (-\infty, \infty)$ be measurable and $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions on E such that*

$$f_n \to f \qquad \text{a.e. on } E.$$

Then for any $\eta > 0$ there is a (measurable) subset S of E with $m(S) < \eta$ such that

$$f_n \to f \qquad \text{uniformly on } E \setminus S.$$

Remark. Again, the condition $m(E) < \infty$ cannot be dropped. Otherwise $f_n = \mathcal{X}_{[n,\infty)}$ and $f = 0$ would be a counter example.

PROOF. We claim that for any $\varepsilon > 0$ and $\delta > 0$, there exists $A \subseteq E$ with $m(A) < \delta$ and $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{whenever } n \geq N \text{ and } x \in E \setminus A.$$

Be careful the above statement is not saying that $f_n \to f$ uniformly on $E \setminus A$ since A depends on ε and δ.

To prove our claim, we let

$$G_n = \{\, x \in E \colon\ |f_n(x) - f(x)| \geq \varepsilon \,\} \qquad \text{(the trouble makers!)}$$

and

$$G = \overline{\lim}\, G_n := \bigcap_{n \in \mathbb{N}} E_n, \qquad \text{where } E_n = \bigcup_{k \geq n} G_k.$$

Note that if $x \in G$ then $x \in E_n$ for all $n \in \mathbb{N}$, it follows that $f_n(x) \nrightarrow f(x)$. Since the set of all x such that $f_n(x) \nrightarrow f(x)$ is of measure zero, we have $m(G) = 0$. Note also that $m(E_1) < \infty$ and E_n "decreases" to G, so $\lim m(E_n) = m(G) = 0$ by Lemma 1.19. There is $N \in \mathbb{N}$ such that $m(E_N) < \delta$. This N, together with $A := E_N$, proved our claim.

Now, let $\eta > 0$ be given. Apply the above result to $\varepsilon = 1/k$ and $\delta = \eta/2^k$, we obtain A_k with $m(A_k) < \eta/2^k$ and $N_k \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{1}{k} \qquad \text{whenever } n \geq N_k \text{ and } x \in E \setminus A_k.$$

Let $S = \bigcup_{k \in \mathbb{N}} A_k$, then $m(S) \leq \sum_{k=1}^{\infty} m(A_k) < \eta$ and $|f_n(x) - f(x)| < 1/k$ whenever $n \geq N_k$ and $x \in E \setminus S$. Hence, $f_n \to f$ uniformly on $E \setminus S$. $\square$