CONVERGENCE AND LIMITS

1. Introduction

In “Introductory Theory of Differentiation” we introduced the theory of limits of functions. In particular we defined the meaning of the phrase

$$\lim_{x\to a} f(x) = L .$$

In this article we will pursue this matter further, where we will study one-sided limits (left-hand and right-hand limits) of functions, as well as limits at infinity and limits of sequences.

2. What is a function?

To avoid putting too much into one single article, in “Introductory Theory of Differentiation” we made no attempts to clarify what a function *is*. We just let the reader interpret the word “function” in whatever way he/she likes. Now to make our discussion precise, let us first clarify what we mean by a function. Let D be a collection of real numbers. For example, D may be the set of all non-zero real numbers, or D may be the interval [0, 1] that consists of the collection of all real numbers x satisfying $0 \le x \le 1$. A real-valued function f defined on D is a rule that assigns one and only one real number to each real number in D. For example,

$$f(x) = \frac{\sin x}{x}$$

is a real-valued function defined on the set of all non-zero real numbers (note that it is not defined at the real number $x = 0$ because division by 0 is meaningless), while

$$g(x) = \frac{x^2 - 4}{x - 2}$$

is a real-valued function defined on the set of all real numbers that are not equal to 2. Also

$$h(x) = \sqrt{x}$$

is a real-valued function defined on the set of all non-negative real numbers (note that the square root of a negative real number is undefined). If f is a real-valued function defined on D, then we call D the **domain** of the function f.

The above definition makes rigorous what we mean by “functions”. We can now proceed to discuss limits of functions.

3. More on Limits of Functions

Recall that, intuitively, by

$$\lim_{x\to a} f(x) = L,$$

we mean that $f(x)$ can be made very close to L whenever x is very close to a. More precisely, this means that we can make $f(x)$ *arbitrarily close* to L (i.e. as close to L as we want) by simply taking x to be *sufficiently close* to a. To talk about whether $f(x)$ is *close* enough to L or whether x is *close* enough to a, it is desirable that we can introduce a quantity to *measure* how close two real numbers are. To do so, let us introduce here the concept of a **distance**.

Definition 3.1.

Let a and b be two real numbers. Then the **distance** $d(a, b)$ between a and b is defined to be the absolute value of their difference, i.e.

$$d(a,b) = |a-b|.$$

This distance has a number of useful properties:

(1) $d(a,b) \geq 0$ for any real numbers a and b, with $d(a,b) = 0$ if and only if $a = b$.

(2) $d(a,b) = d(b,a)$ for any real numbers a and b.

(3) $d(a,c) \leq d(a,b) + d(b,c)$ for any real numbers a, b and c.

The last property (3) is commonly known as the **triangle inequality**. (Compare with the one given in Section 3 of “Introductory Theory of Differentiation”!) Can you prove these three properties?

Now that we have a distance function, we can quantify how close two real numbers are. This paves the way towards defining the meaning of the phrase

$$\lim_{x\to a} f(x) = L.$$

Just one more remark for now: If we consider the real-valued function

$$h(x) = \sqrt{x}$$

that is only defined on the set of non-negative real numbers, it certainly makes no sense to talk about the limit of h as x tends to –1. The reason is that –1 is too *far away* from D. We want to take care of this when we define limits of functions. This leads us to introduce the concept of a **limit point**.

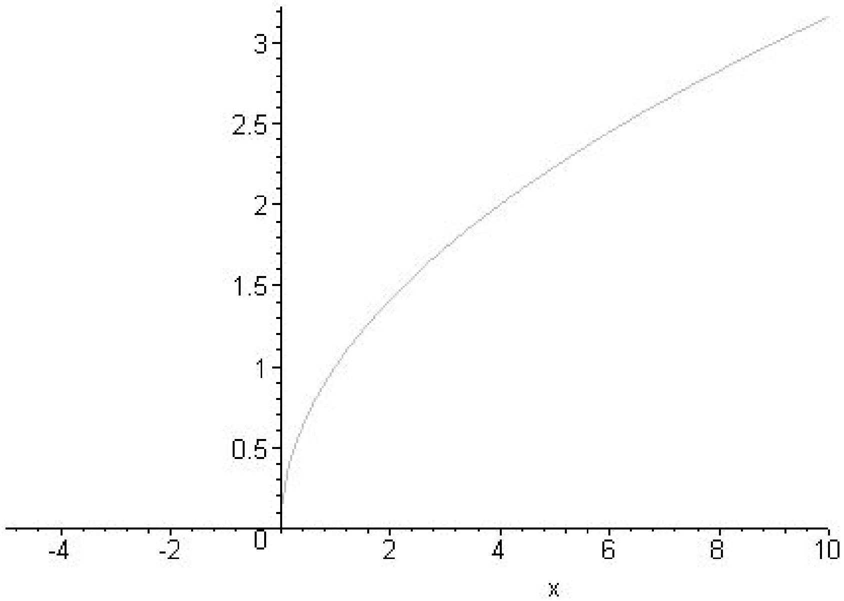

Figure 1: Graph of $f(x) = \sqrt{x}$

Definition 3.2.

Let D be a collection of real numbers. We say a real number a is a **limit point** of D if for any $\delta > 0$, there exists a real number x in D that satisfies $x \neq a$ and $d(x, a) < \delta$.

For example, if D is the set of all non-zero real numbers, then 0 is a limit point of D even though 0 is not in D. Indeed every real number is a limit point of this D. If now D is the set of all positive real numbers, then any non-negative real number is a limit point of D, while any negative number is not a limit point of D. In fact the limit points of D are precisely those real numbers that are *arbitrarily close* to D.

Together with the discussion at the beginning of the Section (also see the discussion in Section 3 of "Introductory Theory of Differentiation"), we are thus led to the following definition: (Compare with the Definition 3.1 of "Introductory Theory of Differentiation"!)

Definition 3.3.

Let D be a collection of real numbers and let f be a real-valued function defined on D. Suppose that a is a limit point of D and L is a real number. We say "$f(x)$ tends to L as x tends to a", or equivalently

$$\lim_{x \to a} f(x) = L,$$

if and only if the following holds:

For any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever x is a real number in D that satisfies $x \neq a$ and $d(x, a) < \delta$, we have

$$d(f(x), L) < \varepsilon.$$

(Note that we have required that x to be in D so that $f(x)$ is defined.)

Does the above definition make sense for you? (It is important that it *does*, because then you understand *why* it is such defined.)

Definition 3.4.

Let D be a collection of real numbers and let f be a real-valued function defined on D. Suppose that a is a limit point of D. If there is a real number L such that

$$\lim_{x \to a} f(x) = L,$$

then we say that

$$\lim_{x \to a} f(x)$$

exists (and is equal to L).

Can you give an example where $\lim_{x \to a} f(x)$ does not exist?

Example 3.1.

Show that for any real number a,

$$\lim_{x \to a} |x|$$

exists and equals $|a|$.

Solution.

First, observe that the function

$$x \mapsto |x|$$

is defined for any real number *x*, so the domain of this function is the set of all real numbers. In particular, any real number *a* is a limit point of the domain of this function. Next, we divide our proof into a number of cases:

Case 1: $a > 0$

For any $\varepsilon > 0$, we can take $\delta = \min(\varepsilon, a) > 0$, and then we will have, for any real number *x* that satisfies $d(x, a) < \delta$, that

$$\big||x| - |a|\big| = |x - a| < \delta \le \varepsilon .$$

This proves, in this case, that

$$\lim_{x \to a} |x|$$

exists and equals $|a|$.

Case 2: $a = 0$

For any real number *x*, we have $\big||x| - |a|\big| = \big||x| - 0\big| = |x|$, so for any $\varepsilon > 0$, we can take $\delta = \varepsilon > 0$, and then we will have, for any real number *x* that satisfies $d(x, a) < \delta$, that

$$\big||x| - |a|\big| = |x| < \delta = \varepsilon .$$

As a result, in this case, we also have that

$$\lim_{x \to a} |x|$$

exists and equals $|a|$.

Case 3: $a < 0$

For any $\varepsilon > 0$, we can take $\delta = \min(\varepsilon, -a) > 0$, and then we will have, for any real number *x* that satisfies $d(x, a) < \delta$, that

$$\big||x| - |a|\big| = |-x + a| = |x - a| < \delta \le \varepsilon .$$

This proves, in this case, that

$$\lim_{x \to a} |x|$$

again exists and equals $|a|$. This completes our proof.

Alternative Solution.

Observe that for any real numbers *x* and *a*, we have

$$\big||x| - |a|\big| \le |x - a|$$

(because $|x|-|a| \le |x-a|$ and $|a|-|x| \le |x-a|$ both hold by Triangle Inequality). Hence for any $\varepsilon >$ 0, we can take $\delta = \varepsilon > 0$, and then we will have, for any real number x that satisfies $d(x, a) < \delta$, that

$$||x|-|a|| \le |x-a| < \delta = \varepsilon .$$

This also completes our proof.

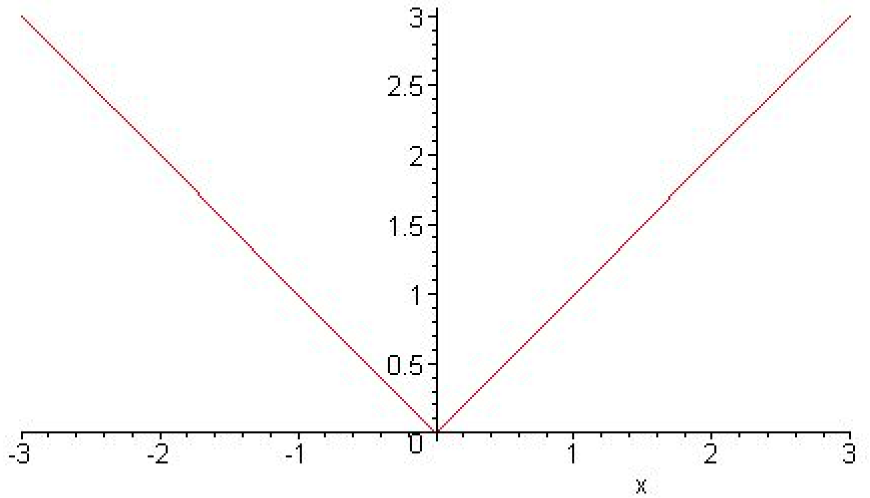

Figure 2: Graph of $f(x)=|x|$

When we write

$$\lim_{x\to a} f(x) = L,$$

we want to be certain that there is no other real number M such that

$$\lim_{x\to a} f(x) = M$$

also holds. This is the content of the following Proposition.

Proposition 3.1.

Let D be a collection of real numbers and let f be a real-valued function defined on D. Suppose that a is a limit point of D and L, M are real numbers. If $f(x)$ tends to both L and M as x tends to a, then $L = M$. In other words, the limit of a function is unique (if it exists).

Proof. Suppose on the contrary that $L \ne M$. Then $|L-M| > 0$, so take

$$\varepsilon = \frac{|L-M|}{2}$$

we have $\varepsilon > 0$, and hence (by definition that $f(x)$ tends to both L and M as x tends to a), we have that there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that:

(I) whenever x is a real number in D that satisfies $x \neq a$ and $d(x,a) < \delta_1$, we have $d(f(x),L) < \varepsilon$;

(II) whenever x is a real number in D that satisfies $x \neq a$ and $d(x,a) < \delta_2$, we have $d(f(x),M) < \varepsilon$.

Now set $\delta = \min(\delta_1, \delta_2)$, we have $\delta > 0$, so taking x_0 in D that satisfies $d(x_0, a) < \delta$ (this is possible because a is a limit point of D), we have, by (I) and (II), that

$$d(f(x_0),L) < \varepsilon \text{ and } d(f(x_0),M) < \varepsilon .$$

This proves that

$$|L - M| = d(L,M) \leq d(f(x_0),L) + d(f(x_0),M) < \varepsilon + \varepsilon = 2\varepsilon = |L - M|,$$

which is a contradiction. Hence $L = M$, as desired.

Q.E.D.

Given two real-valued functions f and g that are defined on the same domain D, we can define their sum $f + g$ and their product $f \cdot g$ by

$$(f + g)(x) \triangleq f(x) + g(x) \text{ and } (f \cdot g)(x) \triangleq f(x) \cdot g(x)$$

for all x in D. Then $f + g$ and $f \cdot g$ are again real-valued functions defined on D. We can also define the quotient function

$$\frac{f}{g}$$

by

$$\frac{f}{g}(x) \triangleq \frac{f(x)}{g(x)}$$

for all x in D', where D' is the set of all x in D that satisfies $g(x) \neq 0$. Then the quotient function is a real-valued function defined on D'.

In the next Proposition we are concerned with the limits of the sums, products and quotients of two functions. Its proof is basically the same as that of Proposition 3.2 in “Introductory Theory of Differentiation”; only minor modifications are called for. It is a good exercise that you reproduce their proofs using mainly the properties (1) to (3) of the distance d introduced above. Try it out!

Proposition 3.2.

Let D be a collection of real numbers and let f, g be real-valued functions defined on D. Suppose that a is a limit point of D. If

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

both exists, then:

1. $\lim_{x\to a}\left(f+g\right)(x)$ exists and is equal to $\lim_{x\to a} f(x) + \lim_{x\to a} g(x)$;
2. $\lim_{x\to a}\left(f\cdot g\right)(x)$ exists and is equal to $\left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right)$; and
3. If

$$\lim_{x\to a} g(x) \neq 0,$$

then there exists $\delta > 0$ such that $g(x) \neq 0$ for all x in D that satisfies $d(x, a) < \delta$, so if we denote by D' the set of all x in D that satisfies $g(x) \neq 0$, then a is a limit point of D',

$$\lim_{x\to a} \frac{f}{g}(x)$$

exists and is equal to

$$\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

Recall that we have

$$\lim_{x\to a} x = a$$

and

$$\lim_{x\to a} c = c$$

for any real constant c. Thus from Proposition 3.2 we obtain:

Corollary 3.3.

For any polynomial $p(x)$ and any real number a, we have

$$\lim_{x\to a} p(x) = p(a).$$

This says that we can evaluate limits of **polynomials** by direct substitution. (But note that in general limits CANNOT be evaluated by direct substitution!)

Next we state and prove the Sandwich Theorem, which is often so useful.

Proposition 3.4. (Sandwich Theorem)

Let D be a collection of real numbers and let f, g, h be real-valued functions defined on D. Suppose that a is a limit point of D, and suppose that there exists $\delta > 0$ such that

$$f(x) \le g(x) \le h(x)$$

whenever x is a real number in D that satisfies $x \ne a$ and $d(x, a) < \delta$. If

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} h(x)$$

both exists and have a common value of L, then

$$\lim_{x \to a} g(x)$$

also exists and is equal to L.

Proof. Let $\varepsilon > 0$ be given. Then since

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} h(x)$$

both exists and are equal to L, we have that there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that:

(I) whenever x is a real number in D that satisfies $x \ne a$ and $d(x,a) < \delta_1$, we have $|f(x) - L| < \varepsilon$; in particular for any such x we have $L - \varepsilon < f(x)$.

(II) whenever x is a real number in D that satisfies $x \ne a$ and $d(x,a) < \delta_2$, we have $|h(x) - L| < \varepsilon$; in particular for any such x we have $h(x) < L + \varepsilon$.

Hence setting $\delta = \min(\delta_1, \delta_2)$, we have $\delta > 0$, and whenever x is a real number in D that satisfies $x \ne a$ and $d(x, a) < \delta$, we have, by (I) and (II), that

$$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon .$$

In particular for any such x we have

$$|g(x) - L| < \varepsilon .$$

This proves that

$$\lim_{x \to a} g(x)$$

exists and is equal to L, and we are done.

Q.E.D.

Example 3.2.

For each real number x, let

$$f(x)=\begin{cases} x & \text{if } x=\dfrac{m}{n} \text{ for some positive integers } m \text{ and } n \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\lim_{x\to 0} f(x)$$

exists and find its value.

Solution.

Simply observe that the inequalities

$$0\le f(x)\le |x|$$

hold for any real number x. Thus from the fact that

$$\lim_{x\to 0} 0 \quad \text{and} \quad \lim_{x\to 0}|x|$$

both exist and equal 0 we conclude, using the Sandwich Theorem, that

$$\lim_{x\to 0} f(x)$$

exists and is equal to 0.

Exercise

Throughout the exercise, suppose that f is a real-valued function and a is a limit point of the domain of definition of f. Also assume L is a real number.

1. Show that the following limits exist and find their values.

   (a) $\lim_{x\to 0}\dfrac{x^2+3}{x-5}$ (b) $\lim_{x\to 2}\dfrac{x^2-4}{x-2}$ (c) $\lim_{x\to 5}\dfrac{x-5}{x^2-25}$

2. Show that if

   $$\lim_{x\to a} f(x)$$

   exists and is equal to L, then

   $$\lim_{x\to a}|f(x)|$$

exists and is equal to $|L|$. Does the converse hold?

3. Show that the following statements are equivalent:
   (a) $\lim_{x\to a} f(x) = 0$
   (b) $\lim_{x\to a} |f(x)| = 0$

4. Assume $f(a)$ is defined. Show that if

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = L,$$

then

$$\lim_{x\to a} f(x)$$

exists and equals $f(a)$.

5. Suppose that the only property that you know of the sine function is that $|\sin x| \le 1$ for all real numbers x. Prove that

$$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0.$$

6. (a) Let a be a real number. If $0 \le a < \varepsilon$ holds for all positive numbers ε, show that $a = 0$. What if we only know $0 \le a \le \varepsilon$ for all positive numbers ε? Does the same conclusion still hold?
   (b) Let a and b be real numbers. If $0 \le d(a,b) \le \varepsilon$ holds for all positive numbers ε, show that $a = b$.

7. Suppose that

$$\lim_{x\to a} f(x) = L.$$

Show that if $L > 0$, then there exists $\delta > 0$ such that

$$f(x) > 0$$

holds for all x in the domain of definition of f that satisfies $x \ne a$ and $d(x,a) < \delta$. Also show, by means of a counter-example, that if we only have $L \ge 0$, then the above conclusion does not hold even with $f(x) > 0$ replaced by $f(x) \ge 0$.

4. One-sided Limits

In this section we look at how left-hand limits and right-hand limits may be defined. We shall only work on left-hand limits; it should then be clear how we can define right-hand limits.

Intuitively, by

$$\lim_{x \to a-} f(x) = L$$

we mean that $f(x)$ can be made arbitrarily close to L as long as we take x to be strictly smaller than a and at the same time sufficiently close to a. This leads us to the following definition of **left limit points** and **left-hand limits**.

Definition 4.1.

Let D be a collection of real numbers. We say a real number a is a **left limit point** of D if for any $\delta > 0$, there exists a real number x in D that satisfies $a - \delta < x < a$.

Note that a left limit point of D is always a limit point of D, but not vice versa.

Definition 4.2.

Let D be a collection of real numbers and let f be a real-valued function defined on D. Suppose that a is a left limit point of D and L is a real number. We say "$f(x)$ tends to L as x tends to a–", or equivalently

$$\lim_{x \to a-} f(x) = L,$$

if and only if the following holds:

For any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever x is a real number in D that satisfies $a - \delta < x < a$, we have

$$d(f(x), L) < \varepsilon.$$

Just like the case for limits, left-hand limits may not always exist. We have the following definition:

Definition 4.3.

Let D be a collection of real numbers and let f be a real-valued function defined on D. Suppose that a is a left limit point of D. If there is a real number L such that

$$\lim_{x\to a-} f(x) = L,$$

then we say that the left-hand limit of f exists at a.

Example 4.1.

For each real number x, let $[x]$ denote the greatest integer smaller than or equal to x. If a is a real number, show that

$$\lim_{x\to a-} \left(x-[x]\right)$$

exists and find its value.

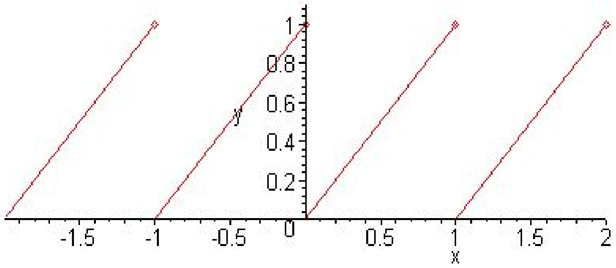

Figure 3: Graph of $f(x) = x-[x]$

Solution.

We will prove that for any real number a,

$$\lim_{x\to a-} \left(x-[x]\right) = \begin{cases} 1 & \text{if } a \text{ is an integer} \\ a-[a] & \text{if } a \text{ is not an integer.} \end{cases}$$

<u>Case 1:</u> a is an integer.

Then for any real number x that satisfies $a-1 \le x < a$, we have $[x] = a - 1$, so for any such x we have

$$d\left(x-[x],\ 1\right) = d\left(x-a+1,\ 1\right) = d(x,a).$$

Hence given any $\varepsilon > 0$, we can simply take $\delta = \min(\varepsilon, 1) > 0$, and then we will have, for any real x that satisfies $a-\delta < x < a$, that $a-1 \le x < a$, so $d\left(x-[x],\ 1\right) = d(x,a) < \varepsilon$. This proves that in this case

$$\lim_{x\to a-} \left(x-[x]\right)$$

exists and is equal to 1.

<u>Case 2:</u> a is not an integer.

Then $a - [a] > 0$, and for any real number x that satisfies $[a] \le x < a$, we have $[x] = [a]$, so for any such x we have

$$d\left(x-[x],\ a-[a]\right) = d\left(x-[a],\ a-[a]\right) = d(x,a).$$

Hence given any $\varepsilon > 0$, we can take $\delta = \min\left(\varepsilon,\ a-[a]\right) > 0$, and then we will have, for any real x that satisfies $a-\delta < x < a$, that $[a] \le x < a$, so $d\left(x-[x],\ a-[a]\right) = d(x,a) < \varepsilon$. This proves that in this case

$$\lim_{x \to a-} \left(x-[x]\right)$$

also exists and is equal to $a - [a]$.

Left-hand limits have properties similar to those of ordinary limits. In particular, Propositions 3.1, 3.3 and 3.4 hold for left-hand limits with trivial modifications. Can you work them out?

Now that we can define left limit points and left-hand limits, we can similarly define right limit points and right-hand limits. Right-hand limits enjoy properties similar to left-hand limits. The link between the three types of limits we introduced so far is given in the following Proposition:

Proposition 4.1.

Let D be a collection of real numbers and let f be a real-valued function defined on D. Suppose that a is a left limit point of D as well as a right limit point of D. Then for any real number L, we have

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = L.$$

Proof. Suppose that

$$\lim_{x \to a} f(x) = L.$$

Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that whenever x is a real number in D that satisfies $x \ne a$ and $|x-a| < \delta$, we have $d\left(f(x), L\right) < \varepsilon$. Hence if x is a real number in D that satisfies $a-\delta < x < a$, we have $d\left(f(x), L\right) < \varepsilon$; if x is a real number in D that satisfies $a < x < a+\delta$, we also have $d\left(f(x), L\right) < \varepsilon$. This proves that

$$\lim_{x \to a-} f(x) = L = \lim_{x \to a+} f(x).$$

Conversely, suppose that

$$\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = L\,.$$

Let $\varepsilon > 0$ be given. Then there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that:

(I) whenever x is a real number in D that satisfies $a-\delta_1 < x < a$, we have $d\left(f(x), L\right) < \varepsilon$; and

(II) whenever x is a real number in D that satisfies $a < x < a+\delta_2$, we have $d\left(f(x), L\right) < \varepsilon$.

Hence setting $\delta = \min(\delta_1, \delta_2)$, we have $\delta > 0$, and whenever x is a real number in D that satisfies $x \neq a$ and $d(x, a) < \delta$, we have, by (I) and (II), that

$$d\left(f(x), L\right) < \varepsilon\,.$$

This proves that

$$\lim_{x\to a} f(x) = L\,.$$

Q.E.D.

The above Proposition sometimes helps us determine whether a given limit

$$\lim_{x\to a} f(x)$$

exists. This is illustrated in the following Example. (An example where the above Proposition does not help is given in the Exercise at the end of this section. See Question 1 there.)

Example 4.2.

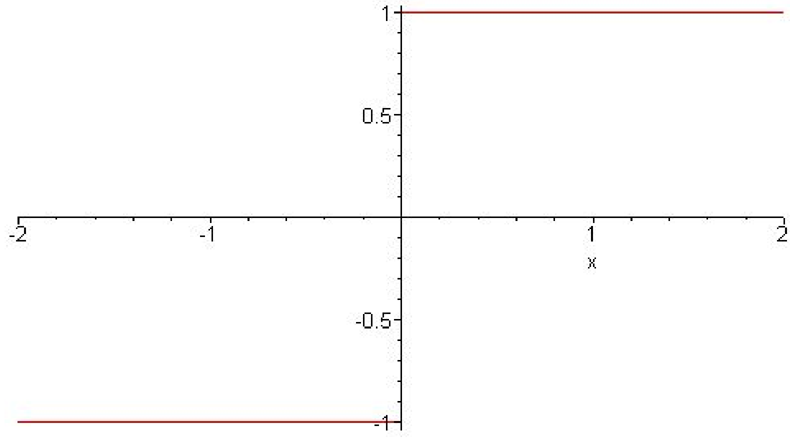

Figure 4: Graph of $f(x) = \dfrac{|x|}{x}$

Show that the limit

$$\lim_{x\to 0} \frac{|x|}{x}$$

does not exist.

Solution.

It is easy to see that the function

$$x \mapsto \frac{|x|}{x}$$

is defined for all non-zero real numbers. So 0 is both a left limit point and a right limit point of the domain of this function. Now

$$\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

so

$$\lim_{x \to 0-} \frac{|x|}{x} = -1 \quad \text{and} \quad \lim_{x \to 0+} \frac{|x|}{x} = 1 .$$

Since the left-hand and right-hand limits are not equal, we conclude that

$$\lim_{x \to 0} \frac{|x|}{x}$$

does not exist.

Note that in Proposition 4.1, it was assumed that *a* is a left limit point of *D* as well as a right limit point of *D*, where *D* is the domain of the function being considered. What if this does not hold? For instance, what if *a* is only a left limit point of *D* but not a right limit point of *D*? The answer is actually simple: if *a* is only a left limit point of *D* but not a right limit point of *D*, then

$$\lim_{x \to a} f(x)$$

exists if and only if

$$\lim_{x \to a-} f(x)$$

exists. Can you prove this?

Exercise

1. Let *g* be the real-valued function defined on the set of all real numbers by

$$g(x) = \begin{cases} 1 & \text{if } \dfrac{1}{x} \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\lim_{x \to 0} g(x)$ does not exist. Explain why it is not helpful here to consider left-hand and right-hand limits.

(Hint: Show that no real numbers can be the limit of $g(x)$ as x tends to zero by definition of limit. Consideration of left-hand and right-hand limits is not helpful here because it is not the "jump" of the function that causes the non-existence of limit. Rather it is the oscillations in the function that leads to the non-existence of limit.)

2. Discuss the existence of the limit $\lim_{x \to a}\left(x - [x]\right)$ for various values of a.

5. Limits at Infinity

Sometimes we wish to discuss the behaviour of a function $f(x)$ as x tends to infinity. This is done as follows.

Definition 5.1.

Let D be a collection of real numbers. We say positive infinity ($+\infty$) is a **limit point** of D if for any real number A, there exists a real number x in D that satisfies $x > A$.

Definition 5.2.

Let D be a collection of real numbers and let f be a real-valued function defined on D. Suppose that $+\infty$ is a limit point of D and L is a real number. We say "$f(x)$ tends to L as x tends to $+\infty$", or equivalently

$$\lim_{x \to +\infty} f(x) = L,$$

if and only if the following holds:

For any $\varepsilon > 0$, there exists a real number A such that whenever x is a real number in D that satisfies $x > A$, we have

$$d(f(x), L) < \varepsilon.$$

The point here is that as long as x is sufficiently large (i.e. sufficiently close to $+\infty$), we will have $f(x)$ arbitrarily close to the limit L. Similarly we can discuss the limit of $f(x)$ as x tends to negative infinity. We sometimes drop the + sign in front of $+\infty$ and simply write ∞ for positive infinity, especially when no confusion will occur. (This will happen in Section 7.)

Note how the above definition resembles the definition of limit that we gave in Sections 3 and 4. Propositions 3.1, 3.2 and 3.4 still hold (under the evident modifications). Can you work out how?

Example 5.1.

Show that

$$\lim_{x \to +\infty} \frac{1}{x} = 0 .$$

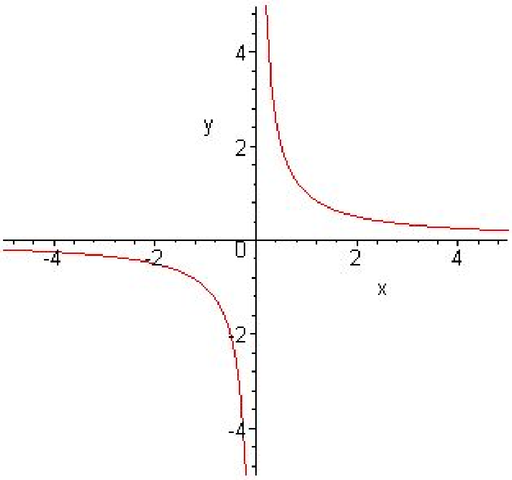

Figure 5: Graph of $f(x) = \frac{1}{x}$

Solution.

It is easy to see that the function

$$x \mapsto \frac{1}{x}$$

is defined for all non-zero real numbers. So $+\infty$ is a limit point of the domain of this function. Now given $\varepsilon > 0$, we can simply take $A = 1/\varepsilon$. Then for any non-zero real number x that satisfies $x > A$, we have

$$d\left(\frac{1}{x}, 0\right) = \frac{1}{x} < \frac{1}{A} = \varepsilon .$$

(Note we used $A > 0$ here.) Hence

$$\lim_{x \to +\infty} \frac{1}{x} = 0 .$$

Example 5.2.

Compute

$$\lim_{x\to+\infty}\frac{x}{x-1}.$$

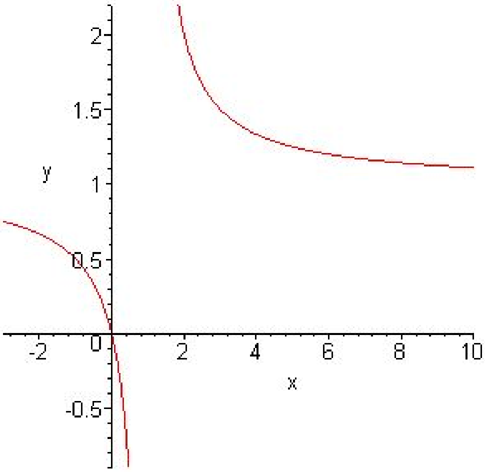

Figure 6: Graph of $f(x)=\dfrac{x}{x-1}$

Solution.

It is easy to see that the function

$$x\mapsto\frac{x}{x-1}$$

is defined for all $x\neq 1$. So $+\infty$ is a limit point of the domain of this function. Now the crucial observation is that

$$\frac{x}{x-1}=\frac{1}{1-\frac{1}{x}}\approx\frac{1}{1-0}=1$$

when x is close to $+\infty$. So using the estimate

$$\left|\frac{x}{x-1}-1\right|=\frac{1}{x-1}$$

which holds for $x > 1$, we see that given any $\varepsilon > 0$, we can simply take $A=1+\dfrac{1}{\varepsilon}$. Then for any real number x that satisfies $x > A$, we have

$$\left|\frac{x}{x-1}-1\right|=\frac{1}{x-1}<\frac{1}{A-1}=\varepsilon.$$

This proves

$$\lim_{x\to+\infty}\frac{x}{x-1}=1.$$

Alternative Solution.

From

$$\frac{x}{x-1}=\frac{1}{1-\frac{1}{x}}$$

which holds for $x \neq 0$ we have, from a variation of Proposition 3.2 and from the previous Example, that

$$\lim_{x\to+\infty}\frac{x}{x-1}$$

exists and that

$$\lim_{x\to+\infty}\frac{x}{x-1}=\frac{1}{1-0}=1.$$

6. Infinite Limits

It is sometimes useful too to be able to give

$$\lim_{x\to a} f(x)=+\infty$$

a precise meaning. We do so in the case where *a* is a real number (and *a* is a limit point of the domain of definition of *f*). The case where *a* is positive or negative infinity is left to you.

Definition 6.1.

Let *D* be a collection of real numbers and let *f* be a real-valued function defined on *D*. Suppose that a real number *a* is a limit point of *D*. We say "*f* (*x*) tends to $+\infty$ as *x* tends to *a*", or equivalently

$$\lim_{x\to a} f(x)=+\infty\,,$$

if and only if the following holds:

For any $M > 0$, there exists $\delta > 0$ such that whenever *x* is a real number in *D* that satisfies $x \neq a$ and $d(x, a) < \delta$, we have

$$f(x) > M\ .$$

The point here is that *f* (*x*) is arbitrarily large as long as *x* is sufficiently close to *a*. Can you work out some examples for yourself?

Exercise

1. Suppose that f is a real-valued function and a is a limit point of the domain of definition of f. Suppose that

$$\lim_{x \to a} f(x) = +\infty .$$

Show that

$$\lim_{x \to a} \frac{1}{f(x)} = 0 .$$

7. Limits of Sequences

Finally we consider a special type of functions, namely those whose domain of definition is the set of positive integers.

Definition 7.1.

Let x be a real-valued function defined on the set of positive integers. Then we say x is a sequence of real numbers. Usually we think of $x(n)$ as the n-th term of the sequence, and we write x_n instead of $x(n)$. The sequence x is also written as $\{x_n\}$.

For example,

$$x_n = \frac{1}{n}$$

defines a sequence of real numbers.

It is obvious that for a function defined on the set of positive integers, the only limit point of its domain of definition is $+\infty$. Thus the definition in Section 5 applies. We state it here in a way that is equivalent to, but that looks slightly different from, the definition given in Section 5.

Definition 7.2.

Let $\{x_n\}$ be a sequence of real numbers and L be a real number. We say "x_n tends to L as n tends to infinity", or equivalently

$$\lim_{n\to\infty} x_n = L,$$

if and only if the following holds:

For any $\varepsilon > 0$, there exists a positive integer N such that whenever n is a positive integer that satisfies $n > N$, we have

$$d(x_n, L) < \varepsilon.$$

Definition 7.3.

Let $\{x_n\}$ be a sequence of real numbers. If there is a real number L such that

$$\lim_{n\to\infty} x_n = L,$$

then we say that

$$\lim_{n\to\infty} x_n$$

exists (and is equal to L). We also say that the sequence $\{x_n\}$ converges (or is convergent). If a sequence does not converge, we say that it diverges (or is divergent).

Again variants of Propositions 3.1, 3.2 and 3.4 hold. In particular, the limit of a sequence of real numbers is unique (if it exists at all); the sum and product of two convergent sequences are convergent; the quotient of two convergent sequences is also convergent provided that the limit of the sequence in the denominator is non-zero; and the Sandwich Theorem holds. To emphasize these theorems, we state them here in the apparently new setting of sequences. Can you supply their proofs?

Proposition 7.1.

Let $\{x_n\}$ be a sequence of real numbers. Suppose that L and M are real numbers. If x_n tends to both L and M as n tends to infinity, then $L = M$. In other words, the limit of a sequence is unique (if it exists at all).

Proposition 7.2.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers. If

$$\lim_{n\to\infty} x_n \quad \text{and} \quad \lim_{n\to\infty} y_n$$

both exists, then:

1. $\lim_{x\to a}(x_n + y_n)$ exists and is equal to $\lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$;
2. $\lim_{x\to a}(x_n y_n)$ exists and is equal to $\left(\lim_{n\to\infty} x_n\right)\cdot\left(\lim_{n\to\infty} y_n\right)$; and
3. If $\lim_{n\to\infty} y_n \neq 0$, then there exists a positive integer N such that $y_n \neq 0$ for all positive integers $n > N$, so $\frac{x_n}{y_n}$ is a real number whenever $n > N$. In this case we actually have $\lim_{n\to\infty}\frac{x_n}{y_n}$ exists and is equal to

$$\frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}.$$

Proposition 7.3. (Sandwich Theorem)

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be three sequences of real numbers and L be a real number. Suppose that there exists a positive integer N such that

$$x_n \le y_n \le z_n$$

for all positive integers $n > N$. If

$$\lim_{n\to\infty} x_n \qquad \text{and} \qquad \lim_{n\to\infty} z_n$$

both exists and have a common value of L, then

$$\lim_{n\to\infty} y_n$$

also exists and is equal to L.

It is "well-known" that

$$\lim_{n\to\infty}\frac{1}{n} = 0.$$

This can be proved, but it involves a deep assumption on the collection of real numbers. So we will simply assume it here and prove the following results.

Example 7.1.

Prove that for any $-1 < a < 1$, we have

$$\lim_{n\to\infty} a^n = 0.$$

Solution.

We divide the proof into three steps:

Step 1: $0 < a < 1$

Since in this case

$$\frac{1}{a} > 1,$$

we can write

$$a = \frac{1}{1+x}$$

for some $x > 0$. Then for any positive integer n, we have, by binomial theorem, that

$$(1+x)^n \geq 1 + nx,$$

and thus

$$0 \leq a^n = \frac{1}{(1+x)^n} \leq \frac{1}{1+nx} = \frac{\frac{1}{n}}{\frac{1}{n} + x}.$$

By Proposition 7.2, as n tends to infinity, the right hand side tends to

$$\frac{0}{0+x} = 0.$$

(Note we used $\lim_{n\to\infty} \frac{1}{n} = 0$ here.) This proves, by Sandwich Theorem, that

$$\lim_{n\to\infty} a^n$$

exists and is equal to 0.

Step 2: $a = 0$

Then $a^n = 0$ for all positive integers n, so clearly

$$\lim_{n\to\infty} a^n = 0.$$

Step 3: $-1 < a < 1$

Then for any positive integer n, we have

$$-|a|^n \leq a^n \leq |a|^n$$

where $0 < |a| < 1$. Thus by Step 1, we have both

$$\lim_{n\to\infty} |a|^n = 0 = \lim_{n\to\infty} \left(-|a|^n\right).$$

This proves, by Sandwich Theorem, that

$$\lim_{n\to\infty} a^n$$

exists and is equal to 0.

Example 7.2.

Prove that for any $a > 0$, we have

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1.$$

Solution.

We divide the proof into two cases:

Case 1: $a \geq 1$

Since in this case

$$a^{\frac{1}{n}} \geq 1$$

for all positive integers *n*, we can define a non-negative sequence $\{x_n\}$ by

$$a^{\frac{1}{n}} = 1 + x_n$$

Then for any positive integer *n*, we have, by binomial theorem, that

$$a = (1 + x_n)^n \geq 1 + nx_n,$$

and thus

$$0 \leq x_n \leq \frac{a-1}{n}.$$

By Proposition 7.2, as *n* tends to infinity, the right hand side tends to 0. (Note we used $\lim_{n\to\infty} \frac{1}{n} = 0$ here.) This proves, by Sandwich Theorem, that

$$\lim_{n\to\infty} x_n$$

exists and is equal to 0. By Proposition 7.2 again, we have that

$$\lim_{n\to\infty} a^{\frac{1}{n}}$$

exists and is equal to 1 + 0 = 1. This completes our proof in this case.

Case 2: $0 < a < 1$

Then

$$\frac{1}{a} > 1$$

so by Case 1, we have

$$\lim_{n\to\infty}\left(\frac{1}{a}\right)^{\frac{1}{n}} = 1.$$

This says

$$\lim_{n\to\infty} a^{-\frac{1}{n}} = 1,$$

and from

$$a^{\frac{1}{n}} = \frac{1}{a^{-\frac{1}{n}}}$$

we have, by Proposition 7.2, that

$$\lim_{n\to\infty} a^{\frac{1}{n}}$$

exists and is equal to $\frac{1}{1} = 1$.

Exercise

Throughout the exercise, suppose that *f* is a real-valued function and *a* is a limit point of the domain of definition of *f*. Also assume *L* is a real number.

1. Determine whether each of the following limit exists or not. Find the limit, with justification, if the limit exists; explain why the limit fails to exist otherwise.

   (a) $\lim_{x\to 3}\frac{1}{1+x}$ (b) $\lim_{x\to 1}\frac{(x-1)^2}{|x-1|}$ (c) $\lim_{x\to 3}\frac{x-3}{x^2-9}$

   (d) $\lim_{x\to 0}\frac{x^2+3x}{|x|}$ (e) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x+x^2}\right)$ (f) $\lim_{x\to +\infty}\left(\frac{1}{x}-\frac{1}{x+1}\right)$

2. Show that the following sequences converge and find their limits. (You may assume that

$$\lim_{n\to\infty}\frac{1}{n} = 0,$$

so you may use the results in Examples 7.1 and 7.2 where appropriate.)

(a) $\lim_{n\to\infty}\frac{2n}{n+3}$ (b) $\lim_{n\to\infty}\frac{(-1)^n n}{n^2-n+2}$ (c) $\lim_{n\to\infty}\frac{n^2}{n!}$

(d) $\lim_{n\to\infty}\frac{3^n}{n!}$ (Hint: For $n>3$, $0<\frac{3^n}{n!}\le 6\left(\frac{3}{4}\right)^n$.)

3. Let $\{x_n\}$ be a sequence of real numbers. Show that the following are equivalent:

(a) $\{x_n\}$ diverges

(b) For any real number l, there exists $\varepsilon>0$ such that for any positive integer N, there exists a positive integer $n>N$ such that $d(x_n,l)\ge\varepsilon$.

(Make sure you can do the above negation properly even if you are not given the answer!)

4. Use Question 3 to show that $\left\{(-1)^n\right\}$ and $\left\{\sin\frac{n\pi}{2}\right\}$ are divergent sequences.

5. Show that if f is a real-valued function defined on the set of all real numbers and if

$$\lim_{x\to+\infty} f(x)=L,$$

then for any sequence $\{x_n\}$ of real numbers that tends to infinity, we have

$$\lim_{n\to\infty} f(x_n)=L.$$

In particular we have

$$\lim_{n\to\infty} f(n)=L.$$

6. Show that the converse to Question 5 holds, namely that: If f is a real-valued function defined on the set of all real numbers and if

$$\lim_{n\to\infty} f(x_n)=L$$

for any sequence $\{x_n\}$ of real numbers that tends to infinity, then

$$\lim_{x\to+\infty} f(x)=L.$$

7. Show that if

$$\lim_{x\to a} f(x)=L$$

then

$$\lim_{x \to a+1} f(x-1) = L .$$

Does the converse hold? Can you interpret this graphically?

8. Show that if

$$\lim_{x \to a} f(x) = L$$

then

$$\lim_{x \to 2a} f\left(\frac{x}{2}\right) = L .$$

Does the converse hold? Can you interpret this graphically? How far can you generalize the above result?

9. Prove that if

$$\lim_{n \to \infty} x_n = L$$

then

$$\lim_{n \to \infty} x_{n+1} = L .$$

(Hint: Consider $\{x_{n+1}\}$ as a new sequence and argue by definition of limit.) Can you see any connection between Questions 7, 8 and the above?

10. Let $\{x_n\}$ be a sequence of non-negative real numbers. Show that if $\{x_n\}$ is convergent, then

$$\lim_{n \to \infty} x_n \geq 0 .$$

Show that it is NOT true that if $\{x_n\}$ is a convergent sequence of positive real numbers, then

$$\lim_{n \to \infty} x_n > 0 .$$

Can you generalize this to the setting of limit of functions?